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Behaviour of wavepackets of the 'Dirac oscillator': Dirac representation versus Foldy–Wouthuysen representation

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Abstract. The behaviour of wavepackets of the 'Dirac oscillator' in $(1 + 1)$ dimensions is examined. The Dirac oscillator is a relativistic extension of the harmonic oscillator. We construct a wavepacket in the usual Dirac representation and another one in the Foldy–Wouthuysen (FW) representation. In the nonrelativistic limit, these wavepackets are both reduced to Schrödinger's coherent wavepacket for the harmonic oscillator. The wavepacket in the Dirac representation exhibits Zitterbewegung whereas the one in the FW representation does not. We find that the wavepacket in the FW representation behaves much more like a classical particle than that of the Dirac representation. This gives insights into Ehrenfest's theorem for the Dirac equation.

1. Introduction

Consider a wavepacket in nonrelativistic quantum mechanics. If the size of the wavepacket is negligibly small, it follows from Ehrenfest's theorem that the expectation values of the coordinate and momentum obey the equation of motion of classical mechanics [1, 2]. If the potential is linear or quadratic in coordinate, the centroid of the wavepacket exactly satisfies the corresponding classical equation of motion even when the size of the wavepacket is not negligible. Do similar situations exist in relativistic quantum mechanics? We try to find insights into this question by examining the behaviour of wavepackets for the one-dimensional version of the 'Dirac oscillator' (DO) [3].

The DO is one of the rare examples such that the solutions of the stationary Dirac equation are all given in simple analytical forms. In the nonrelativistic limit, the DO is reduced to the usual harmonic oscillator (HO). Solutions of the time-dependent Dirac equation for the DO can be written down explicitly, albeit in the form of a series. In a recent paper we examined the time-dependent behaviour of a wavepacket for the DO in detail [4]. The wavepacket of [4] was designed such that it starts as a minimum uncertainty wavepacket that is displaced from the origin. This wavepacket is a relativistic extension of Schrödinger's coherent wavepacket for the nonrelativistic HO [2, 5]. Between the DO and HO wavepackets there are interesting differences which we will mention in due course. Here let us single out the following difference: the DO wavepacket exhibits 'Zitterbewegung' [6] which makes it difficult to relate the wavepacket to its classical counterpart. When

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the wavepacket is expanded as a linear combination of the DO stationary states, negative energy states inevitably enter. The Zitterbewegung is caused by the interference between the positive and negative energy components of the expansion.

In [4] we used the usual representation which we will call the Dirac representation. There is, however, a representation in which Zitterbewegung has no place. This is the one obtained by Foldy and Wouthuysen (FW) in which the positive and negative energy states are completely separated [7]. Historically such a representation was first obtained by Pryce and later independently by Tani [8]. For a free particle, if coordinate x (in one dimension) of the FW representation is interpreted as the position operator for the particle, the relation among x, the velocity $\mathrm{d}x/\mathrm{d}t$, momentum p and energy E take the same form as in classical mechanics. No Zitterbewegung appears in the FW representation. The x of the FW representation agrees with the position operator that was introduced earlier by Newton and Wigner [9] on very general grounds. As was recently emphasized by Costella and McKellar [10], the FW representation (or the Newton–Wigner representation) is the only one in which one can take a meaningful classical limit. Foldy and Wouthuysen worked out the transformation between the Dirac and the FW representations in a closed form only for a free particle but its extension to the case with an interaction is possible.

The purpose of this paper is to construct wavepackets in the Dirac and FW representations for the one-dimensional DO and compare the time-dependent behaviour of these wavepackets. We call the wavepackets in the two representations the Dirac and FW wavepackets, respectively. In section 2 we summarize relevant features of the DO. In section 3 we construct the Dirac and FW wavepackets. Then we present explicit illustrations which give us insights into the difference between the Dirac and FW representations. Summary and discussions are given in section 4.

2. The Dirac oscillator: Dirac versus FW representations

The DO in one dimension is defined by the Dirac Hamiltonian

$$H_{\mathrm{D}} = c\alpha\pi + \beta mc^2 \qquad \pi = p - \mathrm{i}\beta m\omega x \tag{2.1}$$

where $\omega(>0)$ is a constant. For the Dirac matrices we use the 2×2 Pauli matrices, $\alpha = \sigma_y$ and $\beta = \sigma_z$. Heisenberg's equation for x reads as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[H_{\mathrm{D}}, x] = c\alpha. \tag{2.2}$$

This leads to the well known conundrums that obscure the relationship between the relativistic quantum and classical mechanics. Equation (2.2) implies that the measured magnitude of the velocity is always the speed of light, c, leading to the notion of Zitterbewegung [6], interpretation of which is not easy [11]. Secondly, the equation gives no immediate connection between the velocity and momentum. This is in contrast to $\mathrm{d}x/\mathrm{d}t = p/m$ that we obtain in the nonrelativistic case. This is probably the reason why Ehrenfest's theorem for the Dirac equation is seldom discussed.

In their seminal paper, FW showed how the above conundrums can be avoided in the FW representation [7, 8]. For the DO, the FW and Dirac representations are related by the FW transformation [12],

$$U = e^{\mathrm{i}S} \qquad \mathrm{i}S = \frac{1}{2g}\beta(\alpha\pi)\tan^{-1}\left(\frac{g}{mc}\right) \tag{2.3}$$

$$\mathrm{g}^2 = (\alpha\pi)^2 = \pi^\dagger\pi = p^2 + (m\omega x)^2 - \beta\hbar m\omega. \tag{2.4}$$

The FW Hamiltonian is

$$H_{\mathrm{FW}} = UH_{\mathrm{D}}U^{\dagger} = \beta c[(mc)^2 + p^2 + (m\omega x)^2 - \beta\hbar m\omega]^{\frac{1}{2}}. \tag{2.5}$$

The positive energy sector ($\beta = 1$) is completely decoupled from the negative energy sector ($\beta = -1$) in the FW representation.

In the FW representation, Heisenberg's equation for x and p reads as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[H_{\mathrm{FW}}, x] = \frac{\partial H_{\mathrm{FW}}}{\partial p} \tag{2.6}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[H_{\mathrm{FW}}, p] = -\frac{\partial H_{\mathrm{FW}}}{\partial x}. \tag{2.7}$$

Since x and p do not commute, $\partial/\partial p$ and $\partial/\partial x$ in the above are not simple derivatives. In the limit of $\hbar \to 0$, however, these become simple derivatives. Equations (2.6) and (2.7) for the positive energy sector ($\beta = 1$) become the same in form as the canonical equations of motion of classical mechanics. The classical counterparts of equations (2.6) and (2.7) for the negative energy sector ($\beta = -1$) do not make sense. Ehrenfest's theorem is obtained by taking the expectation values of equations (2.6) and (2.7). If we go back to the Dirac representation, the x of the FW representation becomes X such that

$$X = U^{\dagger}xU \tag{2.8}$$

which is different from x: similarly $U^{\dagger}pU \neq p$.

The H_{D} and H_{FW} of course share the same eigenvalues and are given by

$$E_{+(n)} = mc^2\left(1 + \frac{2n\hbar\omega}{mc^2}\right)^{\frac{1}{2}} \tag{2.9}$$

$$E_{-(n)} = -E_{+(n+1)} = -mc^2\left[1 + \frac{2(n+1)\hbar\omega}{mc^2}\right]^{\frac{1}{2}} \tag{2.10}$$

where $n = 0, 1, 2, \ldots$. Note that $E = -mc^2$ is excluded in equation (2.10). Apart from $E_{+(0)} = mc^2$, there is symmetry between the positive and negative energies.

The eigenstates of H_{D} with $E_{+(n)}$ and $E_{-(n)}$ are given by

$$\phi_{\mathrm{D}+(n)} = \begin{pmatrix} A_n\chi_n(x) \\ B_n\chi_{n-1}(x) \end{pmatrix} \tag{2.11}$$

$$\phi_{\mathrm{D}-(n)} = \begin{pmatrix} B_{n+1}\chi_{n+1}(x) \\ -A_{n+1}\chi_n(x) \end{pmatrix} \tag{2.12}$$

$$A_n = \left(\frac{E_n + mc^2}{2E_n}\right)^{\frac{1}{2}} \qquad B_n = \left(\frac{E_n - mc^2}{2E_n}\right)^{\frac{1}{2}} \tag{2.13}$$

where E_n is the $E_{+(n)}$ of equation (2.9) and $\chi_n(x)$ is the usual normalized wavefunction for the nonrelativistic harmonic oscillator. In particular, $\chi_0(x)$ is given by

$$\chi_0(x) = (m\omega/\pi\hbar)^{\frac{1}{4}}\exp(-m\omega x^2/2\hbar). \tag{2.14}$$

For $n = 0$, we find $A_0 = 1$ and $B_0 = 0$.

The eigenstates of H_{FW} with $E_{+(n)}$ and $E_{-(n)}$ are simply given by

$$\phi_{\mathrm{FW}+(n)} = \begin{pmatrix} \chi_n(x) \\ 0 \end{pmatrix} \tag{2.15}$$

$$\phi_{\mathrm{FW}-(n)} = \begin{pmatrix} 0 \\ \chi_n(x) \end{pmatrix}. \tag{2.16}$$

The $\phi_{\mathrm{FW}\pm(n)}$ and $\phi_{\mathrm{D}\pm(n)}$ are related by the FW transformation, i.e. $\phi_{\mathrm{FW}\pm(n)} = U\phi_{\mathrm{D}\pm(n)}$. Note that $\phi_{\mathrm{FW}+(0)} = \phi_{\mathrm{D}+(0)}$, but $\phi_{\mathrm{FW}\pm(n)} \neq \phi_{\mathrm{D}\pm(n)}$ for $n \neq 0$. Except for $n = 0$, the expectation value $\langle x^2 \rangle$ turns out to be greater for $\phi_{\mathrm{D}\pm(n)}$ than for $\phi_{\mathrm{FW}\pm(n)}$.

3. Dirac and FW wavepackets

We now construct wavepackets and examine their time-dependent behaviour. Let us start with the Dirac wavepacket which we have already discussed in a recent paper [4]. We seek a wavepacket, $\Psi_{\mathrm{D}}(x, t)$, that satisfies the time-dependent Dirac equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi_{\mathrm{D}}(x,t) = H_{\mathrm{D}}\Psi_{\mathrm{D}}(x,t) \tag{3.1}$$

and the initial condition

$$\Psi_{\mathrm{D}}(x,0) = \begin{pmatrix} \chi_0(x-x_0) \\ 0 \end{pmatrix}. \tag{3.2}$$

The $\Psi_{\mathrm{D}}(x,0)$ represents a Gaussian wavepacket localized around $x = x_0$. This is a minimum-uncertainty wavepacket such that $(\Delta x)^2(\Delta p)^2 = \hbar^2/4$. The $\Psi_{\mathrm{D}}(x,0)$ can be obtained by operating the displacement operator $\exp(-\mathrm{i}px_0/\hbar)$ on $\phi_{\mathrm{D}+(0)}(x)$, the wavefunction of the state with the lowest positive energy.

In expanding $\Psi_{\mathrm{D}}(x,t)$ in terms of the stationary solutions, we need both of $\phi_{\mathrm{D}+(n)}$ and $\phi_{\mathrm{D}-(n)}$. The negative energy states cannot be arbitrarily dropped. As shown in [4] $\Psi_{\mathrm{D}}(x,t)$ is given by

$$\Psi_{\mathrm{D}1}(x,t) = \frac{1}{2}\eta\Sigma_n\frac{\xi^n}{(n!)^{\frac{1}{2}}}\left[\left(1+\frac{mc^2}{E_n}\right)\mathrm{e}^{-\mathrm{i}E_nt/\hbar} + \left(1-\frac{mc^2}{E_n}\right)\mathrm{e}^{\mathrm{i}E_nt/\hbar}\right]\chi_n(x) \tag{3.3}$$

$$\Psi_{\mathrm{D}2}(x,t) = \frac{1}{2}\eta\Sigma_{n>0}\frac{\xi^n}{(n!)^{\frac{1}{2}}}\frac{(2nmc^2\hbar\omega)^{\frac{1}{2}}}{E_n}(\mathrm{e}^{-\mathrm{i}E_nt/\hbar} - \mathrm{e}^{\mathrm{i}E_nt/\hbar})\chi_{n-1}(x) \tag{3.4}$$

$$\eta = \int_{-\infty}^{\infty}\mathrm{d}x\,\chi_0(x)\chi_0(x-x_0) = \mathrm{e}^{-\frac{1}{2}\xi^2} \qquad \xi^2 = \frac{1}{2\hbar}m\omega x_0^2 \tag{3.5}$$

where $\Psi_{\mathrm{D}1}$ and $\Psi_{\mathrm{D}2}$ are the upper and lower components of Ψ_{D}, respectively, and the summation is over $n = 0, 1, 2, \ldots$. Note that $\eta^2\Sigma_n\xi^{2n}/n! = 1$. The expectation value of H_{D} for Ψ_{D} is given by

$$\langle H_{\mathrm{D}}\rangle = mc^2. \tag{3.6}$$

It is counter-intuitive that $\langle H_{\mathrm{D}}\rangle$ is independent of x_0. For the average value of n in Ψ_{D}, we find

$$n_{\mathrm{av}} = \xi^2 = \frac{1}{2\hbar}m\omega x_0^2 \qquad (\Delta n)^2 \equiv (n^2)_{\mathrm{av}} - (n_{\mathrm{av}})^2 = \xi^2 \tag{3.7}$$

which are independent of time.

We are interested in the expectation values $\langle x\rangle$, $\langle p\rangle$ and $v \equiv \mathrm{d}\langle x\rangle/\mathrm{d}t$ for the Dirac wavepacket Ψ_{D}. They are given by

$$\langle x\rangle_{\mathrm{D}} = \frac{1}{2}x_0\eta^2\Sigma_n\frac{\xi^{2n}}{n!}\left[\left(1+\frac{E_n}{E_{n+1}}\right)\cos(E_{n+1}-E_n)t/\hbar + \left(1-\frac{E_n}{E_{n+1}}\right)\cos(E_{n+1}+E_n)t/\hbar\right] \tag{3.8}$$

$$\langle p\rangle_{\rm D} = -\frac{1}{2}m^2\omega x_0\eta^2\Sigma_n\frac{\xi^{2n}}{n!}\left[\left(\frac{1}{E_n}+\frac{1}{E_{n+1}}\right)\sin(E_{n+1}-E_n)t/\hbar - \left(\frac{1}{E_n}-\frac{1}{E_{n+1}}\right)\sin(E_{n+1}+E_n)t/\hbar\right] \tag{3.9}$$

$$v_{\rm D} \equiv \frac{{\rm d}\langle x\rangle_{\rm D}}{{\rm d}t} = c\langle\alpha\rangle_{\rm D} = -m\omega x_0\eta^2\Sigma_n\frac{\xi^{2n}}{n!}\frac{1}{E_{n+1}}[\sin(E_{n+1}-E_n)t/\hbar+\sin(E_{n+1}+E_n)t/\hbar]. \tag{3.10}$$

Each of $\langle x\rangle_{\rm D}$ and $\langle p\rangle_{\rm D}$ of the Dirac wavepacket consists of two parts, one varies in time more rapidly than the other. The rapidly varying part consists of oscillating functions of $(E_{n+1}+E_n)t/\hbar$. These functions, which originate in the interference between positive and negative energy states, cause Zitterbewegung.

For the FW wavepacket we consider the wavefunction $\Psi_{\rm FW}(x,t)$ such that

$$i\hbar\frac{\partial}{\partial t}\Psi_{\rm FW}(x,t) = H_{\rm FW}\Psi_{\rm FW}(x,t) \tag{3.11}$$

$$\Psi_{\rm FW}(x,0) = \begin{pmatrix}\chi_0(x-x_0)\\0\end{pmatrix}. \tag{3.12}$$

The $\Psi_{\rm FW}(x,t)$ is given by

$$\Psi_{\rm FW}(x,t) = \eta\Sigma_n\frac{\xi^n}{(n!)^{\frac{1}{2}}}\phi_{{\rm FW}+(n)}{\rm e}^{-iE_nt/\hbar} = \eta\Sigma_n\frac{\xi^n}{(n!)^{\frac{1}{2}}}\begin{pmatrix}\chi_n(x)\\0\end{pmatrix}{\rm e}^{-iE_nt/\hbar}. \tag{3.13}$$

Let us emphasize that we do not need $\phi_{{\rm FW}-(n)}$ in the expansion. Unlike in $\Psi_{\rm D}(x,t)$, the lower component of $\Psi_{\rm FW}(x,t)$ remain zero.

If we approximate E_n by $E_n = mc^2+n\hbar\omega$, the upper component of $\phi_{\rm FW}(x,t)$ becomes the wavefunction of Schrödinger's coherent state times the phase factor $\exp(-imc^2t/\hbar)$. The $\Psi_{\rm D}$ and $\Psi_{\rm FW}$ obtained above are not related by the FW transformation $\Psi_{\rm FW} = U\Psi_{\rm D}$. This is because the initial conditions (3.2) and (3.12) are such that $\Psi_{\rm FW} = U\Psi_{\rm D} \neq \Psi_{\rm D}$ at $t=0$. The expectation value of $H_{\rm FW}$ for $\Psi_{\rm FW}$ is given by

$$\langle H_{\rm FW}\rangle = \eta^2\Sigma_n\frac{\xi^{2n}}{n!}E_n \tag{3.14}$$

which is greater than $\langle H_{\rm D}\rangle$. If we approximate E_n with $E_n = mc^2+n\hbar\omega$, equation (3.14) becomes $\langle H_{\rm FW}\rangle = mc^2+\frac{1}{2}m\omega^2x_0^2$. For the average value of n for $\Psi_{\rm FW}$ equation (3.7) holds as such.

The expectation values $\langle x\rangle$, $\langle p\rangle$ and ${\rm d}\langle x\rangle/{\rm d}t$ in the FW representation are given by

$$\langle x\rangle_{\rm FW} = x_0\eta^2\Sigma_n\frac{\xi^{2n}}{n!}\cos[(E_{n+1}-E_n)t/\hbar] \tag{3.15}$$

$$\langle p\rangle_{\rm FW} = -m\omega x_0\eta^2\Sigma_n\frac{\xi^{2n}}{n!}\sin[(E_{n+1}-E_n)t/\hbar] \tag{3.16}$$

$$v_{\rm FW} \equiv \frac{{\rm d}\langle x\rangle_{\rm FW}}{{\rm d}t} = -x_0\eta^2\Sigma_n\frac{\xi^{2n}}{n!}[(E_{n+1}-E_n)/\hbar]\sin[(E_{n+1}-E_n)t/\hbar]. \tag{3.17}$$

The $\langle x\rangle_{\rm FW}$ is the $X(t)$ of equation (3.12) of [4]. Note that $\langle x\rangle_{\rm FW}$ and $\langle p\rangle_{\rm FW}$ are free from Zitterbewegung. If we approximate $E_{n+1}-E_n$ with $\hbar\omega$, the $\langle x\rangle_{\rm FW}$ and $\langle p\rangle_{\rm FW}$ are reduced to those of Schrödinger's nonrelativistic coherent state.

We want to compare the behaviour of the wavepackets constructed above with their classical counterpart. Let us define the classical counterpart by means of the FW Hamiltonian for the positive energy sector ($\beta=1$),

$$H_c = c[(mc)^2+p^2+(m\omega x)^2-\hbar m\omega]^{\frac{1}{2}}. \tag{3.18}$$

We retain the term with $\overline{h}$, although it should be dropped in the strictly classical situation. The classical equations of motion with this H_c are

$$\dot{x} = \frac{\partial H_c}{\partial p} = \frac{c^2 p}{E} \qquad \dot{p} = -\frac{\partial H_c}{\partial x} = -\frac{(mc\omega)^2 x}{E} \tag{3.19}$$

where $\dot{x} = \mathrm{d}x/\mathrm{d}t$ and $E = H_c =$ constant. The x and p are given by

$$x(t) = x_0 \cos \omega_p t \qquad \omega_p = \frac{mc^2}{E}\omega \tag{3.20}$$

$$p(t) = \frac{E}{c^2}\dot{x}(t) = -m\omega x_0 \sin \omega_p t \tag{3.21}$$

where we have chosen the initial condition $x(0) = x_0$ and $p(0) = 0$, which corresponds to the initial conditions (3.2) and (3.12) for the wavepackets.

We are now ready to examine the wavepackets and their classical counterpart in detail. Before doing so, however, it would be appropriate to give some general remarks about the wavepackets of the DO. One may expect that the DO wavepackets that we have constructed are very similar to Schrödinger's coherent wavepacket of the nonrelativistic HO [2, 5]. Because the DO energy levels are not equally spaced, however, complications arise in the behaviour of the DO wavepackets. Unlike Schrödinger's coherent state, the shape (in terms of the probability density distribution) of the DO wavepacket (in either of the Dirac and the FW representation) changes in time and the initial wavefunction is not exactly restored. For the wavepackets we have constructed, $(\Delta x)^2(\Delta p)^2 = \overline{h}^2/4$ at $t = 0$ but $(\Delta x)^2(\Delta p)^2 > \overline{h}^2/4$ for $t > 0$. We have already discussed such features for the Dirac wavepacket in detail [4].

Throughout the following numerical illustrations we express the energy such as E and $\overline{h}\omega$ in units of mc^2, x in $\overline{h}/mc$, t in $\overline{h}/mc^2$, p in mc and v in c. For $\overline{h}\omega$, we take $\overline{h}\omega = 0.1$ (i.e. $\overline{h}\omega/mc^2 = 0.1$) in all illustrations. For x_0 we examine two cases, $x_0 = 5$ and $x_0 = 30$. The n_{av} is respectively 1.25 and 45 in these two cases. Figures 1–4 are for $x_0 = 5$ and figures 5–8 are for $x_0 = 30$. Each figure has two parts; part (a) is for the Dirac representation and part (b) for the FW representation. Figure 1 shows the probability density

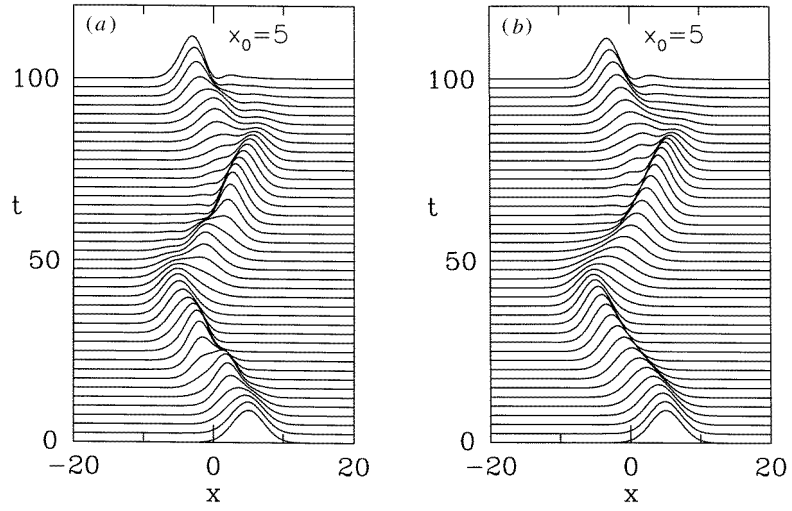

Figure 1. The density $\rho(x, t) = |\Psi(x, t)|^2$ of the wavepacket in an arbitrary scale for (a) the Dirac representation and (b) the FW representation. x is in units of $\overline{h}/mc$ and t in $\overline{h}/mc^2$. The values of the parameters are: $\overline{h}\omega/mc^2 = 0.1$ and $x_0 = 5(\overline{h}/mc)$.

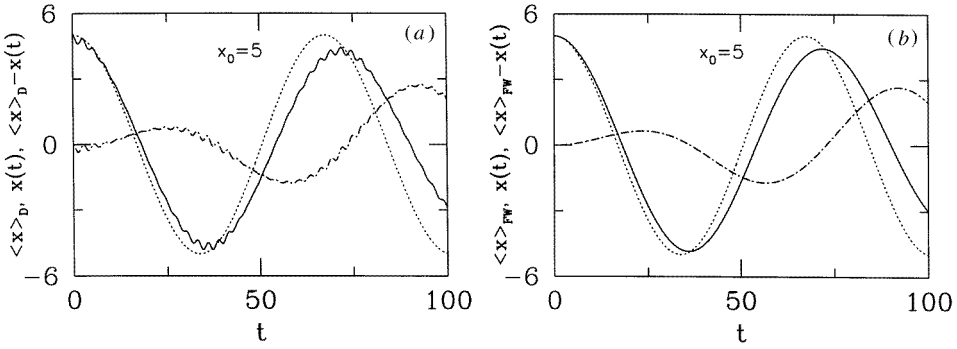

Figure 2. The expectation value, $\langle x\rangle$ (full curve), its classical counterpart, $x(t)$ (dotted curve), and the difference, $\langle x\rangle - x(t)$ (chain curve), are shown for (a) the Dirac representation and (b) the FW representation. x is in units of $\overline{h}/mc$ and t in $\overline{h}/mc^2$. The values of the parameters are the same as those of figure 1.

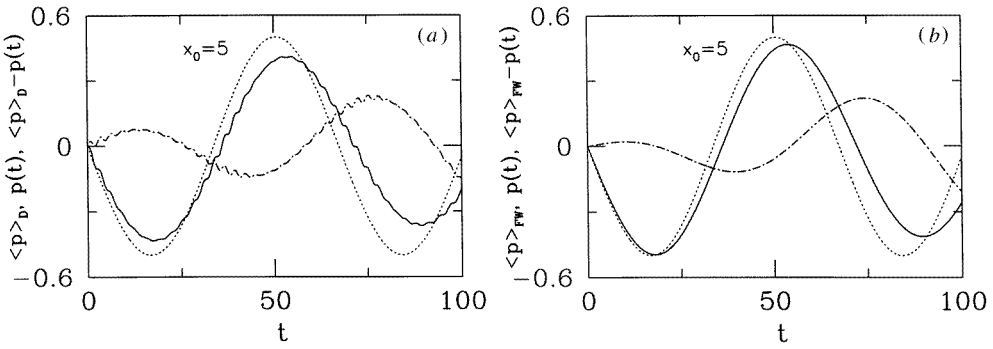

Figure 3. The expectation value, $\langle p\rangle$ (full curve), its classical counterpart, $p(t)$ (dotted curve), and the difference, $\langle p\rangle - p(t)$ (chain curve), are shown for (a) the Dirac representation and (b) the FW representation. p is in units of mc and t in $\overline{h}/mc^2$. The values of the parameters are the same as those of figure 1.

$\rho(x,t) = |\Psi(x,t)|^2$ in an arbitrary scale in the two representations. The density in the FW representation is smoother. Figure 2 compares $\langle x\rangle_{\rm D}$, $\langle x\rangle_{\rm FW}$ and the classical $x(t)$. Similarly figures 3 and 4 compare $\langle p\rangle_{\rm D}$ etc and $v_{\rm D}$, respectively. In the Dirac representation the expectation values exhibit rapid oscillations which we interpret as Zitterbewegung, which is most conspicuous in figure 4.

Figures 5–8 show the same quantities as those of figures 1–4 but $x_0 = 30$ in figures 5–8 as compared with $x_0 = 5$ of figures 1–4. Note the difference in the scales for x and t between figures 1–4 and figures 5–8. The wavepackets of figure 5 oscillate more slowly than those of figure 1. This is because the DO energy separation is smaller for larger values of n. In figures 6 and 7 the amplitudes of Zitterbewegung are so small that they are almost undiscernible.

In figures 2–4, the smooth part of $\langle x\rangle_{\rm D}$ etc are very similar to $\langle x\rangle_{\rm FW}$ etc, which in turn are not very different from $x(t)$ etc, respectively. We expected similar situations in figures 6–8, and is indeed the case in figures 6 and 8, but not in figure 7. Figure 7 is particularly noteworthy. It shows that the smooth part of $\langle p\rangle_{\rm D}$ and $\langle p\rangle_{\rm FW}$ are very different. We carefully confirmed this rather surprising result. The $\langle p\rangle_{\rm FW}$ is much closer to $p(t)$ than

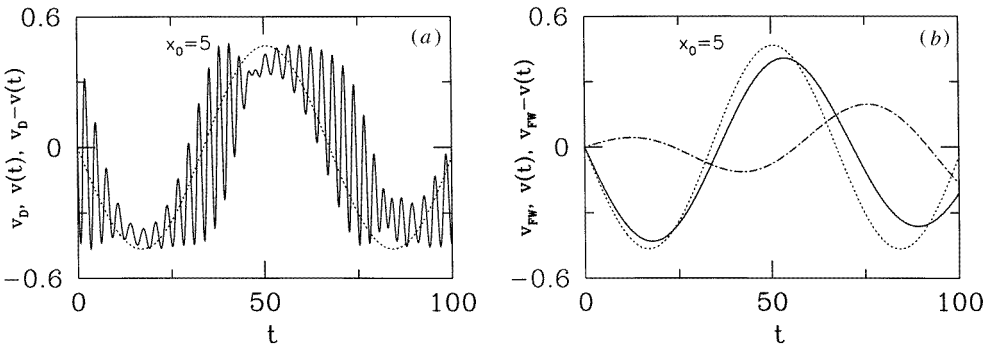

Figure 4. The velocity $v = \mathrm{d}\langle x\rangle/\mathrm{d}t$ (full curve) of the wavepacket, its classical counterpart, $v(t)$ (dotted curve), and the difference between them (chain curve), are shown for (*a*) the Dirac representation and (*b*) the FW representation. The chain curve is not shown in (*a*) because it is very awkward to plot. v is in units of c and t in $\overline{h}/mc^2$. The values of the parameters are the same as those of figure 1.

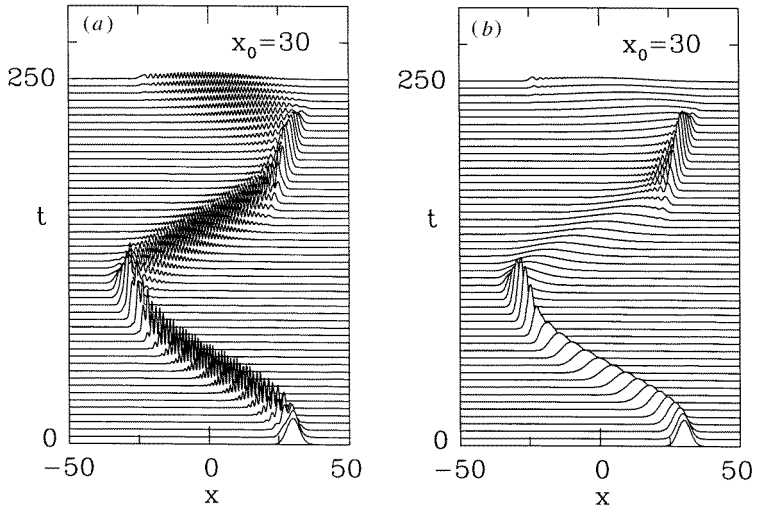

Figure 5. The density $\rho(x,t) = |\Psi(x,t)|^2$ of the wavepacket in an arbitrary scale for (*a*) the Dirac representation and (*b*) the FW representation. The parameters and units are the same as those of figure 1 except that $x_0 = 30$ $(\overline{h}/mc)$.

$\langle p\rangle_{\mathrm{D}}$ is. It is clear that the behaviour of the FW wavepacket is much more like its classical counterpart.

Even the FW wavepacket does not follow the trajectory of its classical counterpart very closely. This is not surprising because $\overline{h} \neq 0$ in the calculation. Recall that n_{av} is 1.25 for figures 1–4 and 45 for figures 5–8. By comparing these two sets of figures, we notice that the gap between the wavepacket and its classical counterpart diminishes when n_{av} is increased. We confirmed this trend by repeating the calculation for even larger values of x_0 (and hence for larger values of n_{av}).

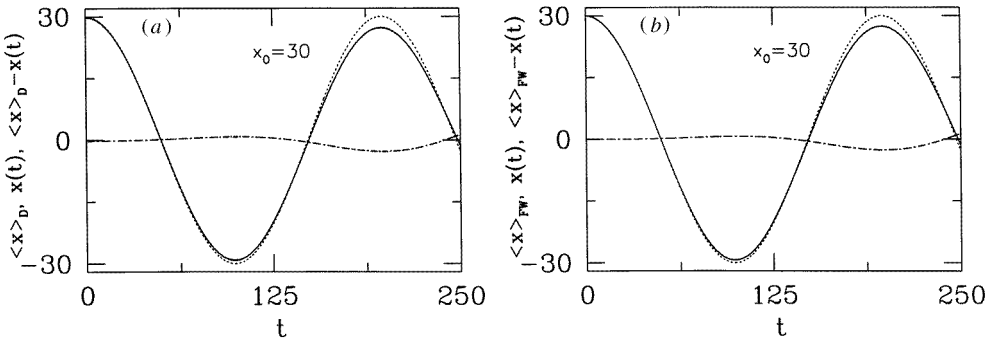

Figure 6. The expectation value, $\langle x\rangle$ (full curve), its classical counterpart, $x(t)$ (dotted curve), and the difference, $\langle x\rangle - x(t)$ (chain curve), are shown for (a) the Dirac representation and (b) the FW representation. The parameters and units are the same as those of figure 2 except that $x_0 = 30(\overline{h}/mc)$.

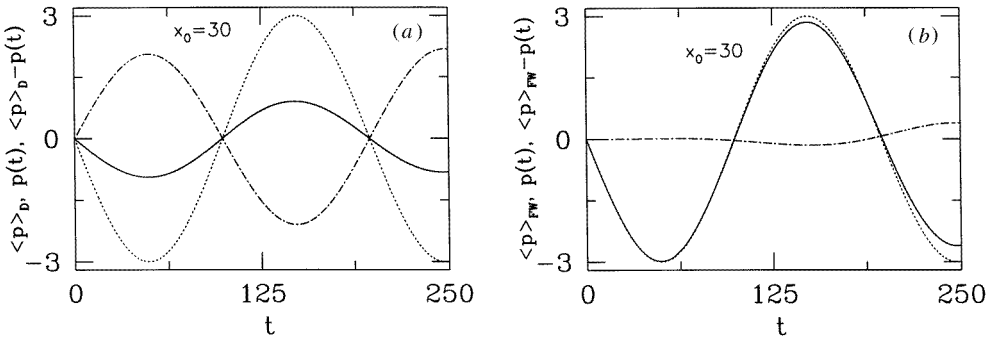

Figure 7. The expectation value, $\langle p\rangle$ (full curve), its classical counterpart, $p(t)$ (dotted curve), and the difference, $\langle p\rangle - p(t)$ (chain curve), are shown for (a) the Dirac representation and (b) the FW representation. The parameters and units are the same as those of figure 3 except that $x_0 = 30(\overline{h}/mc)$.

4. Summary and discussion

For the one-dimensional version of the Dirac oscillator which is exactly solvable, we illustrated the behaviour of wavepackets in the Dirac and FW representations. In the FW representation the positive and negative energy states are completely decoupled. The centroid of the wavepacket exhibits Zitterbewegung in the Dirac representation but not in the FW representation. The wavepacket, although it dissipates in time, behaves much more like a classical particle in the FW representation than in the Dirac representation.

Ehrenfest's theorem can be obtained by taking the expectation values of Heisenberg's equations for x and p. The theorem takes a natural form in the FW representation. In the limit of $\overline{h} \to 0$, Heisenberg's equations for x and p of the positive energy sector ($\beta = 1$) of the FW representation become the same in form as the canonical equation of motion of classical mechanics. In this way Ehrenfest's theorem gives a link between relativistic quantum and classical mechanics. We believe that the features that we have found are of a general nature rather than peculiar to the specific model that we have chosen. We concur with Costella and McKellar [10] in that the FW representation is the only one in which a meaningful classical limit can be obtained.

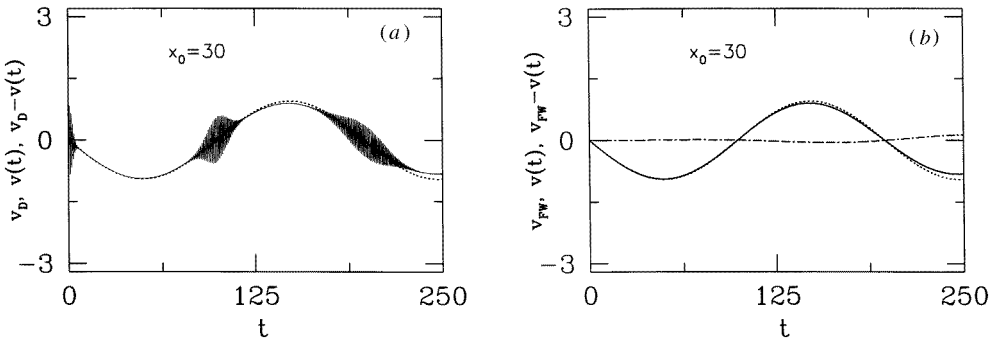

Figure 8. The velocity $v = \mathrm{d}\langle x\rangle/\mathrm{d}t$ (full curve) of the wavepacket, its classical counterpart, $v(t)$ (dotted curve), and the difference between them (chain curve), are shown for (a) the Dirac representation and (b) the FW representation. The chain curve is not shown in (a) because it is very awkward to plot. The parameters and units are the same as those of figure 4 except that $x_0 = 30(\hbar/mc)$.

The wavepackets of the DO that we have examined are relativistic extensions of Schrödinger's coherent wavepacket of the nonrelativistic HO. Unlike the HO, however, the energy levels of the DO are not equally spaced. This makes it impossible to construct a wavepacket that can remain cohesive for a long time. It is, however, possible to modify the DO in such a way that the energy levels become equally spaced. This can be done by replacing $m\omega x$ of equation (2.1) with general 'superpotential' $f(x)$ and determining $f(x)$ by means of the 'inverse scattering' method. This problem is similar to example D of [13]. Then one can have a wavepacket such that, although its shape changes between the recurring times, its initial wavefunction recurs periodically. This wavepacket is not dispersed like those we have examined.

It is generally difficult to find the FW transformation in a closed form. Let us mention two systems for which the FW transformation can be found in a closed form: (a) a charged particle of spin half placed in a magnetic field (but no electric field), and (b) a neutral particle of spin half with an anomalous magnetic moment (like the neutron) placed in an electric field (but no magnetic field). For (a) see [14] for example; for system (b), the Hamiltonian due to the interaction between the magnetic moment μ and the electric field $\boldsymbol{\mathcal{E}}$ is, in standard notation,

$$\tfrac{1}{2}\mu\beta\sigma^{\mu\nu}F_{\mu\nu} = -\mathrm{i}\mu\beta\boldsymbol{\alpha}\cdot\boldsymbol{\mathcal{E}}. \tag{4.1}$$

The Dirac Hamiltonian of the system is

$$H_{\mathrm{D}} = c\boldsymbol{\alpha}\cdot\boldsymbol{\pi} + \beta mc^2 \qquad \boldsymbol{\pi} = \mathbf{p} - \frac{\mathrm{i}}{c}\mu\beta\boldsymbol{\mathcal{E}}. \tag{4.2}$$

If $\boldsymbol{\mathcal{E}}$ is proportional to $\mathbf{r}$, this system is nothing but the DO in three dimensions [3, 12]. If $\boldsymbol{\mathcal{E}}$ is of the form of $(x, 0, 0)$, the system is reduced to the DO in one dimension. Note that $\boldsymbol{\pi}$ commutes with βmc^2. The FW transformation can be worked out in the same way as equations (2.3)–(2.5). This system is of interest in relation to the Aharonov–Casher effect [15]. For extensions of the two cases given above, see [16]. Even when the FW transformation cannot be found in a closed form, it can be worked out in a successive manner as shown by FW. In this sense we can assume the existence of the FW transformation in general.

Finally let us add a remark regarding the Klein–Gordon (KG) equation for the spin zero particle. When appropriately interpreted, the KG equation is an acceptable equation

in quantum mechanics. It has a second-order time derivative, but can be rewritten as a pair of equations which contain only first-order time derivatives. In this new form the wavefunction can be regarded as a two-component one and the Hamiltonian is a 2×2 matrix. This Hamiltonian can be diagonalized by a transformation which is similar to the FW transformation. When this is done the positive and negative energy solutions become decoupled. In each of the positive and negative energy sectors, probability interpretation of the wavefunction is possible [17]. Ehrenfest's theorem for the KG equation can be worked out in a way similar to that for the Dirac equation.

Acknowledgments

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References

[1] Ehrenfest P 1927 *Z. Phys.* **45** 455
[2] Schiff L I 1968 *Quantum Mechanics* 3rd edn (New York: McGraw-Hill) ch 3
[3] Ito D, Mori K and Carriere J 1967 *Nuovo Cimento* A **51** 119
Cook P A 1971 *Lett. Nuovo Cimento* **1** 419
Moshinsky M and Szczepaniak A 1989 *J. Phys. A: Math. Gen.* **22** L817
[4] Nogami Y and Toyama F M 1996 *Can. J. Phys.* **74** 114
[5] Schrödinger E 1926 Die Naturwissenschaften **14** 664
[6] Schrödinger E 1930 Sit. Berlin. Akad. 418
[7] Foldy L L and Wouthuysen S A 1950 *Phys. Rev.* **78** 29
[8] Pryce M H L 1935 *Proc. R. Soc.* A **150** 166
Tani S 1949 Soryushiron Kenkyu **1** 15 (in Japanese)
Tani S 1951 *Prog. Theor. Phys.* **6** 267
[9] Newton T D and Wigner E P 1949 *Rev. Mod. Phys.* **21** 400
[10] Costella J P and McKellar B H J 1995 *Am. J. Phys.* **63** 1119
[11] Koba Z 1956 *Nuovo Cimento* **3** 1 and 214
Koba Z 1959 *Suppl. Prog. Theor. Phys.* **8** 1
Aharonov Y and Bohm D 1957 *Suppl. Nuovo Cimento* **3** 429
[12] Moreno M and Zentella A 1989 *J. Phys. A: Math. Gen.* **22** L821
Beckers J and Debergh N 1990 *Phys. Rev.* D **42** 1255
[13] Nogami Y and Toyama F M 1993 *Phys. Rev.* A **47** 1708
[14] Corinaldesi E and Strocchi F 1963 *Relativistic Wave Mechanics* (Amsterdam: North-Holland) ch 7
de Vries E 1970 *Fortschr. Phys.* **18** 149
Tsai W 1973 *Phys. Rev.* D **7** 1945
[15] Aharonov Y and Casher A 1984 *Phys. Rev. Lett.* **53** 319
Hagen C R 1990 *Phys. Rev. Lett.* **64** 2347
[16] Moreno M, Martínez R and Zentella 1990 *Mod. Phys. Lett.* A **5** 949
[17] Case K M 1954 *Phys. Rev.* **95** 1323
Feshbach H and Villars F 1958 *Rev. Mod. Phys.* **30** 24
Bjorken J D and Drell S D 1964 *Relativistic Quantum Mechanics* (New York: McGraw-Hill) ch 9